Section 1.4 The Matrix Equation $A\mathbf{x} = b$

Definition (Product of a Matrix and a Vector)

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of A and $\mathbf{x}$**, denoted by $A\mathbf{x}$, is **the linear combination of the columns of A using the corresponding entries in x as weights**; that is,

$$A\mathbf{x} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

Remark: $A\mathbf{x}$ is defined only if the number of columns of A equals the number of entries in $\mathbf{x}$.

1 column, 2 entries

Eg: $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not defined.

Example 1. (a) Use the definition to compute

$$\begin{array}{cc} \vec{a}_1 & \vec{a}_2 \end{array}$$

$$\begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} \begin{array}{l} x_1 \\ x_2 \end{array} = x_1\vec{a}_1 + x_2\vec{a}_2$$

$$= 1 \cdot \begin{bmatrix} 6 \\ -4 \\ 7 \end{bmatrix} + (-3)\begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 7 \end{bmatrix} + \begin{bmatrix} -15 \\ 9 \\ -18 \end{bmatrix} = \begin{bmatrix} -9 \\ 5 \\ -11 \end{bmatrix}$$

Row-Vector Rule for Computing $A\mathbf{x}$

If the product $A\mathbf{x}$ is defined, then the i th entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$.

Example 1. (b) Use the Row-Vector Rule to compute the product in part (a).

$$A\vec{x} = \begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \times 1 + 5 \times (-3) \\ (-4) \times 1 + (-3) \times (-3) \\ 7 \times 1 + 6 \times (-3) \end{bmatrix} = \begin{bmatrix} -9 \\ 5 \\ -11 \end{bmatrix}$$

Definition (Matrix Equation)

An equation in the form of $A\mathbf{x} = \mathbf{b}$ is called a **matrix equation**.

Theorem 3 (Equivalence Between Matrix Equation and Vector Equation)

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$$

Example 2. Write the system first as a vector equation and then as a matrix equation.

$$\begin{aligned} 8x_1 - x_2 &= 4 \\ 5x_1 + 4x_2 &= 1 \\ x_1 - 3x_2 &= 2 \end{aligned}$$

ANS: Vector eqn:

$$x_1 \begin{bmatrix} 8 \\ 5 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

Matrix eqn:

$$\begin{bmatrix} 8 & -1 \\ 5 & 4 \\ 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

Existence of Solutions

The definition of $A\mathbf{x}$ leads directly to the following useful fact:

$\boxed{\text{The equation } A\mathbf{x} = \mathbf{b} \text{ has a solution if and only if } \mathbf{b} \text{ is a linear combination of the columns of } A.}$

Theorem 4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

1. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
2. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
3. The columns of A span $\mathbb{R}^m$.
4. A has a pivot position in every row.

Warning: Theorem 4 is about a coefficient matrix, not an augmented matrix. If an augmented matrix $[A \quad \mathbf{b}]$ has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ may or may not be consistent.

Eg: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the augmented matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. has a pivot position in every row. But the system has no solution.

Example 3. Let $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{bmatrix}$. Is $\mathbf{u}$ in the plane in $\mathbb{R}^3$ spanned by the columns of A? (~~See the figure~~.) Why or why not?

ANS: The vector $\vec{u}$ is in the plane spanned by the columns of A if and only if $\vec{u}$ is a linear combination of the columns of A.

This happens if and only if $\underline{A\vec{x} = \vec{u}}$ has a solution (See the box above Thm 4).

The corresponding augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -5 & 0 \\ -2 & 6 & 1 \\ 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ -2 & 6 & 1 \\ 3 & -5 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 8 & 3 \\ 0 & -8 & -3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 8 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & \frac{3}{8} \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & \frac{5}{8} \\ 0 & 1 & \frac{3}{8} \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 = \frac{5}{8} \\ x_2 = \frac{3}{8} \\ 0 = 0 \end{cases}$$

Thus the equation $A\vec{x} = \vec{u}$ has a (unique) solution.

So $\vec{u}$ is in the plane spanned by the columns of A.

If we multiply A by the vector $\vec{x} = \begin{bmatrix} \frac{5}{8} \\ \frac{3}{8} \end{bmatrix}$

Writes $\vec{u}$ as a linear combination of the columns of A. i.e.

$$\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{5}{8}\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + \frac{3}{8}\begin{bmatrix} -5 \\ 6 \\ 1 \end{bmatrix}$$

Example 4. Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$. Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span $\mathbb{R}^3$?

Theorem 4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

1. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
2. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
3. The columns of A span $\mathbb{R}^m$.
4. A has a pivot position in every row.

ANS: By Thm 4

$\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathbb{R}^3 \iff A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$ has a pivot position in each row.

Let

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & -1 & -1 \\ -1 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} -1 & 3 & -3 \\ 0 & -1 & -1 \\ 0 & 0 & \cancel{2}\,1 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 3 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus A has a pivot position in each row. So $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathbb{R}^3$.

Properties of the Matrix-Vector Product $A\mathbf{x}$

Theorem 5

If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:

a. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;

b. $A(c\mathbf{u}) = c(A\mathbf{u})$.

The following three questions are left as exercises. I will provide the complete notes for solving them after the lecture.

Exercise 5. Given A and $\mathbf{b}$, write the augmented matrix for the linear system that corresponds to the matrix equation $A\mathbf{x} = \mathbf{b}$. Then solve the system and write the solution as a vector.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

ANS: To solve $A\mathbf{x} = \mathbf{b}$, row reduce the augmented matrix $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}]$ for the corresponding linear system:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -3 & -1 & 2 & 1 \\ 0 & 5 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 5 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 0 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 5 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -4/5 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3/5 \\ 0 & 1 & 0 & -4/5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The solution is $\begin{cases} x_1 = 3/5 \\ x_2 = -4/5. \\ x_3 = 1 \end{cases}$ As a vector, the solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -4/5 \\ 1 \end{bmatrix}$.

Exercise 6. Let $A = \begin{bmatrix} -3 & -4 \\ 12 & 16 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Show that the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution for some choices of $\mathbf{b}$, and describe the set of all $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ does have a solution.

ANS: The augmented matrix for $A\mathbf{x} = \mathbf{b}$ is $\begin{bmatrix} -3 & -4 & b_1 \\ 12 & 16 & b_2 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} -3 & -4 & b_1 \\ 0 & 0 & b_2 + 4b_1 \end{bmatrix}$. This shows that the equation $A\mathbf{x} = \mathbf{b}$ is not consistent when $b_2 + 4b_1 \neq 0$

The set of $\mathbf{b}$ for which the equation is consistent is a line through the origin - the set of all points (b_1, b_2) satisfying $b_2 = -4b_1$.

Exercise 7. Let $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$, and $\mathbf{z} = \begin{bmatrix} -4 \\ -6 \\ 7 \end{bmatrix}$.

It can be shown that $2\mathbf{x} - 3\mathbf{y} - \mathbf{z} = \mathbf{0}$. Use this fact (and no row operations) to find x_1 and x_2 that satisfy the equation

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 7 \end{bmatrix}$$

ANS: As $2\mathbf{x} - 3\mathbf{y} - \mathbf{z} = \mathbf{0}$, we know $2\mathbf{x} - 3\mathbf{y} = \mathbf{z}$, which is the following vector equation

$$2 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 7 \end{bmatrix}$$

This is equivalent to the matrix equation

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 7 \end{bmatrix}$$

Thus $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.